BMath-II, Rings and Modules: Mid-semestral exam

Instructions: Total time 3 Hours. Solve as many problems as you like, for a max score of 30. Please use only results proved in the class, without proving them again. If you wish to use any problem from any source, please provide its solution.

1. Let R be a factorization domain (i.e. an integral domain in which every nonzero, nonunit element factors into irreducubles). Assume any two elements in R have a gcd. Prove that R is a UFD. (3)

2. Give an example of a UFD R and elements $a, b \in R$ such that $\gcd(a, b)$ is not of the form $\lambda a + \mu b$ for $\lambda, \mu \in R$. (4)

3. Let I be a nonzero ideal in the ring of Gaussian integers $\mathbb{Z}[i]$. Prove that $\mathbb{Z}[i]/I$ is finite. (4)

4. Let $R := \mathbb{Z}[i]$. Compute the number of elements, as well as the characteristics of the rings
(i) $R/\langle 3 \rangle$ (ii) $R/\langle 1+i \rangle$ (iii) $R/\langle 1+2i \rangle$. (6)

5. Let $\alpha = 7+2i$, $\beta = 3-4i$ in $\mathbb{Z}[i]$. Find $\sigma, \rho \in \mathbb{Z}[i]$ such that $N(\rho) < N(\beta)$ and $\alpha = \beta\sigma + \rho$. (2)

6. Let R be any integral domain. Prove that the polynomial ring $R[X]$ has infinitely many monic irredicuble polynomials. (6)

7. Let $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ be the algebra of real quaternion as discussed in the class. Prove that the equation $X^2 = -1$ has infinitely many solutions in $\mathbb{H}$. (6)

8. Let F be a field and let $\mathbb{H}_F := F \oplus Fi \oplus Fj \oplus Fk$, where $i, j, k = ij$ have the usual meaning, denote the quaternions over F, as discussed in the class. Give an example of a field F such that $\mathbb{H}_F$ is not a division ring, explaining your answer. (4)

9. Let R be a UFD and $f(X, Y) \in R[X, Y]$ be the polynomial $f(X, Y) = Y^5 + X^4Y^3 + X^3Y^2 + X$. Is f irreducible in $R[X, Y]$? explain. (2)